Quantum deformations of associative algebras and integrable systems

2009 J. Phys. A: Math. Theor. 42 095201

(http://iopscience.iop.org/1751-8121/42/9/095201)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 095201 (18pp) doi:10.1088/1751-8113/42/9/095201

Quantum deformations of associative algebras and integrable systems

B G Konopelchenko

Dipartimento di Fisica, Universita del Salento, and INFN, Sezione di Lecce, 73100 Lecce, Italy

Received 3 November 2008
Published 4 February 2009
Online at stacks.iop.org/JPhysA/42/095201

Abstract
Quantum deformations of the structure constants for a class of associative noncommutative algebras are studied. It is shown that these deformations are governed by the quantum central systems which have a geometrical meaning of a vanishing Riemann curvature tensor for Christoffel symbols identified with the structure constants. A subclass of isoassociative quantum deformations is described by the oriented associativity equation and, in particular, by the Witten–Dijkgraaf–Verlinde–Verlinde equation. It is demonstrated that a wider class of weakly (non)associative quantum deformations is connected with the integrable soliton equations too. In particular, such deformations for the three-dimensional and infinite-dimensional algebras are described by the Boussinesq equation and KP hierarchy, respectively.

PACS number: 02.30.Ik
Mathematics Subject Classification: 16xx, 35Q53, 37K10, 53Axx

1. Introduction

Modern theory of deformations for associative algebras which was formulated in the classical works by Gerstenhaber [1, 2] got a fresh impetus with the discovery of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation [3, 4]. Beautiful formalization of the theory of the WDVV equation in terms of the Frobenius manifolds given by Dubrovin [5, 6] and its subsequent extension to F-manifolds [7, 8] have provided us with the remarkable realization (see e.g. [5–11]) of one of Gerstenhaber's approaches to the deformation of associative algebras which consists in the treatment of 'the set of structure constants as parameter space for the deformation theory' ([1], Chapter II, section 1). A characteristic feature of the theory of Frobenius and F-manifolds is that the action of the algebra is defined on the tangent sheaf of these manifolds [5–11].

A different method to describe deformations of the structure constants for associative commutative algebra in a given basis has been proposed recently in [12–14]. This approach consists (1) in converting the table of multiplication for an associative algebra in the basis $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{N-1}$, i.e.

$$\mathbf{P}_j\mathbf{P}_k = C^l_{jk}(x)\mathbf{P}_l, \qquad j,k = 0,1,\ldots,N-1 \tag{1.1}$$

into the zero set Γ of the functions

$$f_{jk} = -p_j p_k + C^l_{jk}(x)p_l, \qquad j,k = 0,1,\ldots,N-1 \tag{1.2}$$

with $p_0, p_1, \ldots, p_{N-1}$ and deformation parameters $x^0, x^1, \ldots, x^{N-1}$ being the Darboux canonical coordinates in the symplectic space R^{2N} and (2) in the requirement that the ideal J generated by the functions f_{jk} is the Poisson ideal, i.e.

$$\{J, J\} \subset J \tag{1.3}$$

with respect to the standard Poisson bracket $\{,\}$ in R^{2N}. Here and below the summation over the repeated index is assumed and this index always runs from 0 to $N-1$.

Deformations of the structure constants C^l_{jk} defined by these conditions are governed by the central system (CS) of equations consisting of the associativity condition

$$C^l_{jk}(x)C^n_{lm}(x) - C^l_{mk}(x)C^n_{lj}(x) = 0 \tag{1.4}$$

and the coisotropy condition

$$[C,C]^m_{jklr} \doteqdot C^m_{sj}\frac{\partial C^s_{lr}}{\partial x^k} + C^m_{sk}\frac{\partial C^s_{lr}}{\partial x^j} - C^m_{sr}\frac{\partial C^s_{jk}}{\partial x^l} - C^m_{sl}\frac{\partial C^s_{jk}}{\partial x^r} + C^s_{lr}\frac{\partial C^m_{jk}}{\partial x^s} - C^s_{jk}\frac{\partial C^m_{lr}}{\partial x^s} = 0. \tag{1.5}$$

Such deformations of the structure constants have been called the coisotropic deformations in [12–14]. CS (1.4), (1.5) is invariant under the general transformations $x^j \to \widetilde{x}^j$ of the deformation parameters with C^l_{jk} being the (1,2) type tensor [14]. It has a number of other interesting properties. For the finite-dimensional algebras this CS contains as the particular cases the oriented associativity equation, the WDVV equation and certain hydrodynamical-type equations such as the stationary dispersionless Kadomtsev–Petviashvili (KP) equation (or Khokhlov–Zabolotskaya equation) [14]. For the infinite-dimensional polynomial algebras in the Faa' di Bruno basis the coisotropic deformations are described by the universal Whitham hierarchy of zero genus, in particular, by the dispersionless KP hierarchy [12].

It was demonstrated in [14] that the theory of coisotropic deformations and the theory of F-manifolds are essentially equivalent as far as the characterization (1.4), (1.5) of the structure constants is concerned. One of the advantages of the former is that it is formulated basically in a simple framework of classical mechanics with the standard ingredients such as the phase space with the canonical coordinates p_j, x^j and the constraints $f_{jk} = 0$ which are nothing but the Dirac's first class constraints. This feature of the approach proposed in [12–14] strongly suggests a way to build a natural and simple quantum version of coisotropic deformations in parallel with the passage from the classical to quantum mechanics.

In this paper, we present the basic elements of the theory of quantum deformations for a class of associative noncommutative algebras. A main idea of the approach is to associate the elements of the Heisenberg algebra with the elements P_j of the basis for the algebra and deformation parameters x^j . Realizing the table of multiplication, following the Dirac's prescription, as the set of equations selecting 'physical' subspace in the infinite-dimensional linear space and requiring that this subspace is not empty, one gets a system of equations, the quantum central system (QCS),

$$\hbar\frac{\partial C^n_{jk}}{\partial x^l} - \hbar\frac{\partial C^n_{kl}}{\partial x^j} + C^m_{jk}C^n_{lm} - C^m_{kl}C^n_{jm} = 0 \tag{1.6}$$

which governs quantum deformations of the structure constants. Here $\hbar$ is the Plank's constant.

It is shown that a subclass of isoassociative quantum deformations, for which the classical condition (1.4) is valid for all values of quantum deformation parameters, is described by the oriented associativity equation and, as the reduction, by the WDVV equation. A wider class of weakly (non)associative quantum deformations is considered too. It is characterized by nonvanishing quantum anomaly (defect of associativity). These deformations are also associated with integrable systems. It is shown that for the three-dimensional algebra a class of such deformations is described by the Boussinesq equation. For infinite-dimensional polynomial algebras in the Faa' di Bruno basis the weakly (non)associative quantum deformations of the structure constants are given by the KP hierarchy or, more generally, by the multi-component KP hierarchy.

The paper is organized as follows. The definition of quantum deformations and the derivation of the QCS (1.6) are given in section 2. Isoassociative quantum deformations and the corresponding oriented associativity equation are discussed in section 3. In section 4 the weakly (non)associative deformations and an example of such deformation described by the Boussinesq equation are considered. Quantum deformations of the infinite-dimensional algebra and associated KP hierarchy are studied in section 5.2.

2. Quantum deformations of associative algebras

In the construction of quantum version of the coistropic deformations we will follow basically the same lines as in the standard passage from classical mechanics to quantum mechanics: substitute a phase space by the infinite-dimensional linear (Hilbert) space, introduce operators instead of the canonically conjugated momenta and coordinates etc.

So, let A be an N-dimensional associative algebra with (or without) unity element $\mathbf{P}_0$. We will consider a class of algebras which posses a basis composed by pairwise commuting elements. Denoting elements of a basis as $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{N-1}$ one writes the table of multiplication

$$\mathbf{P}_j\mathbf{P}_k = C^l_{jk}(x)\mathbf{P}_l, \qquad j,k = 0,1,\ldots,N-1, \tag{2.1}$$

where $x^0, x^1, \ldots, x^{N-1}$ stand for the deformation parameters of the structure constants. The commutativity of the elements of the basis implies that $C^l_{jk} = C^l_{kj}$.

In order to define quantum deformations we first associate a set of linear operators $\widehat{p}_j$ and $\widehat{x}^j$ $(j = 0,1,\ldots,N-1)$ with the elements $\mathbf{P}_j$ of the basis and the deformation parameters x^j and require that these operators are elements of the Heisenberg algebra

$$[\widehat{p}_j, \widehat{p}_k] = 0, \qquad [\widehat{x}^j, \widehat{x}^k] = 0, \qquad [\widehat{p}_j, \widehat{x}^k] = \hbar\delta^k_j, \qquad j,k = 0,1\ldots,N-1, \tag{2.2}$$

where $\hbar$ is Planck's constant and δ^k_j is the Kronecker symbol. The second step is to give a realization of the table of multiplication (2.1) in terms of these operators. For this purpose we introduce the set of operators $\widehat{f_{jk}}$ defined by

$$\widehat{f}_{jk} = -\widehat{p}_j\widehat{p}_k + C^l_{jk}(\widehat{x})\widehat{p}_l, \qquad j,k = 0,1,\ldots,N-1. \tag{2.3}$$

To simplify notations we will omit in what follows the label $\widehat{\ }$ in the symbols of operators.

It is easy to see that the representation of the table of multiplication (2.1) by the operator equations $f_{jk} = 0$ is too restrictive. Indeed, it implies the relation $[f_{jk}, p_n] = 0$ which due to the identity

$$\left[p_n, C^l_{jk}\right] = \hbar\frac{\partial C^l_{jk}}{\partial x^n} \tag{2.4}$$

gives $\frac{\partial C^l_{jk}}{\partial x^n} = 0$. A right way to quantize the first-class constraints from the classical mechanics has been suggested long time ago by Dirac [15]. It consists in the treatment of the first-class constraints as the conditions selecting a subspace H_Γ of physical states in the Hilbert space H by the equations

$$f_{jk}\,|\Psi\rangle = 0, \qquad j,k = 0,1,\ldots,N-1, \tag{2.5}$$

where vectors $|\Psi\rangle \subset H$.

The prescription (2.5) is the key point of the following

Definition. *The structure constants $C^l_{jk}(x)$ are said to define quantum deformations of an associative algebra if the operators f_{jk} defined by (2.3) have a nontrivial common kernel.*

If conditions (2.5) are satisfied then any vector $|\Psi\rangle$ belonging to H_Γ is invariant under the group of transformations generated by operators $G = \exp(\alpha_{jk} f_{jk})$ where α_{jk} are parameters, i.e. $G\,|\Psi\rangle = |\Psi\rangle$. In such a form the above definition is the quantum version of the classical condition (1.3).

The requirement (2.5) for the existence of the common eigenvectors with zero eigenvalues for all operators f_{jk} imposes severe constraints on the functions $C^l_{jk}(x)$. We begin with the well-known consequence of (2.5) that is

$$[f_{jk}, f_{\ln}]|\Psi\rangle = 0, \qquad j,k,l,n = 0,1,\ldots,N-1. \tag{2.6}$$

This condition is the quantum version of the coisotropy condition $\{f_{jk}, f_{\ln}\}|_\Gamma = 0$ in the classical case. Using (2.2) and (2.4), one obtains from (2.6) the relation

$$\left(\hbar^2 \frac{\partial^2 C^m_{jk}}{\partial x^l \partial x^n} - \hbar^2 \frac{\partial^2 C^m_{\ln}}{\partial x^j \partial x^k} - \hbar\,[C,C]^m_{jk\ln}\right) p_m\,|\Psi\rangle = 0, \tag{2.7}$$

where the bracket $[C,C]^m_{jk\ln}$ is defined in (1.5).

So, equations (2.6) are satisfied if

$$\hbar \frac{\partial^2 C^m_{jk}}{\partial x^l \partial x^n} - \hbar \frac{\partial^2 C^m_{\ln}}{\partial x^j \partial x^k} - [C,C]^m_{jk\ln} = 0, \qquad j,k,l,n,m = 0,1,\ldots,N-1. \tag{2.8}$$

This constraint is the quantum version of the cosisotropy condition (1.5).

To derive the quantum version of the associativity condition (1.4) we use the identity

$$(p_j p_k)p_l - p_j\,(p_k p_l) = p_j f_{kl} - p_l f_{jk} + C^m_{kl} f_{jm} - C^m_{jk} f_{lm}$$
$$+ \left(\hbar \frac{\partial C^n_{jk}}{\partial x^l} - \hbar \frac{\partial C^n_{kl}}{\partial x^j} + C^m_{jk} C^n_{lm} - C^m_{kl} C^n_{jm}\right) p_n. \tag{2.9}$$

It implies that

$$((p_j p_k)p_l - p_j(p_k p_l))|\Psi\rangle = \left(\hbar \frac{\partial C^n_{jk}}{\partial x^l} - \hbar \frac{\partial C^n_{kl}}{\partial x^j} + C^m_{jk} C^n_{lm} - C^m_{kl} C^n_{jm}\right) p_n|\Psi\rangle \tag{2.10}$$

for $|\Psi\rangle \subset H_\Gamma$. Hence, if the structure constants obey the equations

$$\hbar \frac{\partial C^n_{jk}}{\partial x^l} - \hbar \frac{\partial C^n_{kl}}{\partial x^j} + C^m_{jk} C^n_{lm} - C^m_{kl} C^n_{jm} = 0, \qquad j,k,l,n = 0,1,\ldots,N-1 \tag{2.11}$$

then

$$((p_j p_k)p_l - p_j(p_k p_l))|\Psi\rangle = 0. \tag{2.12}$$

Equations (2.11) and (2.8) represent the quantum counterpart of the classical CS (1.4), (1.5). These system of equations have in fact a much simpler form since only part of them is independent. Indeed, one has the following identity

$$\hbar T^m_{jk,\ln} = \hbar \frac{\partial R^m_{jlk}}{\partial x^n} - \hbar \frac{\partial R^m_{nlk}}{\partial x^j} - C^m_{js} R^s_{lkn} - C^m_{ns} R^s_{kjl} - C^s_{\ln} R^m_{ksj} - C^s_{jk} R^m_{\ln s} - C^s_{lk} R^m_{sjn},$$

where $T^m_{jk,\mathrm{ln}}$ denotes the lhs of equation (2.8) and R^n_{klj} stands for the lhs of equation (2.11). Thus, we have the following proposition.

Proposition 2.1. *The structure constants $C^l_{jk}(x)$ define a quantum deformation of an associative algebra if they obey the equations*

$$R^n_{klj} \doteq \hbar\frac{\partial C^n_{jk}}{\partial x^l} - \hbar\frac{\partial C^n_{kl}}{\partial x^j} + C^m_{jk}C^n_{lm} - C^m_{kl}C^n_{jm} = 0.\square \tag{2.13}$$

We will refer to system (2.13) as the quantum central system (QCS). We emphasize that quantum deformations are defined in the category of associative noncommutative algebras which possess commutative basis.

Proposition 2.2. *If the structure constants define a quantum deformation then*

$$[f_{jk}, f_{lm}] = -\hbar K^{st}_{jk,lm} f_{st}, \qquad j,k,l,m = 0,1,\ldots,N-1, \tag{2.14}$$

where

$$K^{st}_{jk,lm} = \frac{1}{2}\left(\delta^t_m\frac{\partial C^s_{jk}}{\partial x^l} + \delta^t_l\frac{\partial C^s_{jk}}{\partial x^m} - \delta^t_k\frac{\partial C^s_{lm}}{\partial x^j} - \delta^t_j\frac{\partial C^s_{lm}}{\partial x^k}\right.$$
$$\left. + \delta^s_m\frac{\partial C^t_{jk}}{\partial x^l} + \delta^s_l\frac{\partial C^t_{jk}}{\partial x^m} - \delta^s_k\frac{\partial C^t_{lm}}{\partial x^j} - \delta^s_j\frac{\partial C^t_{lm}}{\partial x^k}\right). \tag{2.15}$$

The proof is by direct calculation.

We note that expression (2.15) exactly coincides with that which appear in the coisotropic case for the Poisson brackets between the functions f_{jk}. So, one has the same closed algebra for the basic objects f_{jk} and $\widehat{f}_{jk}$ for the coisotropic and quantum deformations up to the standard correspondence $[,] \longleftrightarrow -\hbar\,\{,\}$ [15] between commutators and Poisson brackets.

The central systems (1.4), (1.5) and (2.13) which define coisotropic and quantum deformations have rather different forms. In spite of this they have some general properties in common. The invariance under the general transformations of deformation parameters is one of them. Similar to the coisotropic case [12–14] the quantum deformation parameters x^j and corresponding p_k are strongly interrelated: they should obey the conditions (2.2). So any change $x^j \to \widetilde{x}^j$ requires an adequate change $p_k \to \widetilde{p}_k$ in order the relations (2.2) to be preserved. Thus, for the general transformation of the deformation parameters x^j in our scheme one has

$$x^j \to \widetilde{x}^j,\ p_k \to \widetilde{p}_k = \frac{\partial\widetilde{x}^n}{\partial x^k}p_n, \qquad j,k = 0,1,\ldots,N-1. \tag{2.16}$$

Note that the transformations (2.16) preserve the commutativity of the basis.

The requirement of the invariance for equations (2.5) readily implies that

$$C^l_{jk}(x) \to \widetilde{C}^l_{jk}(\widetilde{x}) = \frac{\partial\widetilde{x}^l}{\partial x^t}\frac{\partial x^s}{\partial\widetilde{x}^j}\frac{\partial x^m}{\partial\widetilde{x}^k}C^t_{sm}(x) + \hbar\frac{\partial\widetilde{x}^l}{\partial x^m}\frac{\partial^2 x^m}{\partial\widetilde{x}^j\partial\widetilde{x}^k} \tag{2.17}$$

under transformations (2.16). Then, it is a straightforward check that equation (2.13) is also invariant. Hence, one has

Proposition 2.3. *The QCS (2.13) is invariant under the general transformations of the deformation parameters.*

Furthermore, the relation (2.17) evidently coincides with the transformation law of the Christoffel symbols and in formula (2.13) the tensor R^n_{klj} is nothing but the Riemann curvature tensor expressed in terms of the Christoffel symbols (see e.g. [16, 17]).

Thus, we have the following geometrical interpretation.

Geometrical interpretation. The QCS system (2.13) which governs the quantum deformations in geometrical terms means the vanishing of the Riemann curvature tensor R^n_{klj} for the torsionless Christoffel symbols Γ^l_{jk} identified with the structure constants $\left(C^l_{jk} = \hbar\Gamma^l_{jk}\right)$.

In the standard terms of the matrix-valued one-form Γ with the matrix elements (see e.g. [17])

$$\Gamma^l_k = (C_j)^l_k \, \mathrm{d}x^j = C^l_{jk} \, \mathrm{d}x^j \tag{2.18}$$

equation (2.13) looks like

$$\hbar d\Gamma + \Gamma \wedge \Gamma = 0, \tag{2.19}$$

where d and $\wedge$ denote the exterior differential and exterior product, respectively. The flatness condition

$$[\nabla_j, \nabla_l] = 0 \tag{2.20}$$

for the torsionless connection $\nabla_j = \hbar\frac{\partial}{\partial x^j} + C_j$ is the another standard form of equation (2.13). In the context of Frobenius manifolds the relation between the structure constants and Christoffel symbols has been discussed with a different approach in [6].

The identification of the structure constants with the Christoffel symbols leads to certain constraints within such geometrical interpretation. For instance, for an algebra with the unity element $\mathbf{P}_0$, for which $C^l_{0k} = \delta^l_k$, equation (2.13) immediately implies

$$\frac{\partial C^n_{jk}}{\partial x^0} = 0, \qquad j, k, n = 0, 1, \ldots, N-1. \tag{2.21}$$

Furthermore, if one requires that $\mathbf{P}_0$ is invariant with respect to the transformations (2.16) then $\frac{\partial \widetilde{x}^j}{\partial x^0} = \delta^j_0$ and $\frac{\partial x^j}{\partial \widetilde{x}^0} = \delta^j_0$, $j = 0, 1, \ldots, N-1$.

For algebras with different properties (semisimple, nilpotent etc) the orbits generated by transformations (2.16) have quite different parametrizations. For instance, for a semisimple algebra there is a basis at which $C^l_{jk} = \delta_{jk}\delta^l_j$ (see e.g. [5, 6]). Let us denote the deformation parameters associated with this basis as $u^0, u^1, \ldots, u^{N-1}$. Then the corresponding orbit has the following parametrization

$$C^l_{jk}(x) = \frac{\partial x^l}{\partial u^m}\frac{\partial u^m}{\partial x^j}\frac{\partial u^m}{\partial x^k} + \hbar\frac{\partial x^l}{\partial u^m}\frac{\partial^2 u^m}{\partial x^j \partial x^k}, \tag{2.22}$$

where $x^m(u)$, $m = 0, 1, \ldots, N-1$ are arbitrary functions. For a nilpotent algebra for which all elements have degree of nilpotency equal to 2 there exists a basis at which all $C^l_{jk} = 0$. The general element of the corresponding orbit is given by the formula

$$C^l_{jk}(x) = \hbar\frac{\partial x^l}{\partial u^m}\frac{\partial^2 u^m}{\partial x^j \partial x^k}, \tag{2.23}$$

where again $x^m(u)$ are arbitrary functions.

In the construction presented above we did not use concrete realization of operators p_j and x^k. Any such realization provides us with a concrete realization of the associative algebra under consideration and the formulae derived . The most common representation of the Heisenberg algebra (2.2) is given by the so-called Schrödinger representation at which operators $\widehat{x}^j$ are the operators of multiplication by x^j, p_j are operators $\hbar\frac{\partial}{\partial x^j}$ and wavefunctions $\Psi(x)$ are elements of the space H. In this realization the associative algebra A is the well-known algebra of differential polynomials and equations (2.5) have the form

$$-\hbar\frac{\partial^2\Psi}{\partial x^j \partial x^k} + C^l_{jk}(x)\frac{\partial\Psi}{\partial x^l} = 0, \qquad j, k = 0, 1, \ldots, N-1. \tag{2.24}$$

It is a simple check that the usual compatibility condition for system (2.24) (equality of the mixed third-order derivatives) is nothing but conditions (2.12) and it is equivalent to equations (2.11). For an algebra with the unity element one has $\hbar \frac{\partial \Psi}{\partial x^0} = \Psi$ and, in virtue of (2.21) one has

$$\Psi(x) = \mathrm{e}^{\frac{x^0}{\hbar}} \widetilde{\Psi}(x^1, \ldots, x^{N-1}). \tag{2.25}$$

The system (2.24) is well-known in geometry. In the theory of the Frobenius manifolds it is called the Gauss–Manin equation (see e.g. [6, 9]). Such a system arises also in the theory of Gromov–Witten invariants [18, 19].

The standard quasiclassical approximation $\Psi = \exp\left(\frac{S(x)}{\hbar}\right), \hbar \to 0$ (see e.g. [15]) performed for equations (2.24) give rise to the Hamilton–Jacobi equations

$$-\frac{\partial S}{\partial x^j}\frac{\partial S}{\partial x^k} + C^l_{jk}\frac{\partial S}{\partial x^l} = 0. \tag{2.26}$$

These equations coincide with those for the generating function S for Lagrangian submanifolds which arise in the theory of coisotropic deformations [14]. In this classical limit $\hbar \to 0$, the system (2.8), (2.11) is reduced to the classical CS (1.4), (1.5) and the whole construction presented above is reduced to that of coisotropic deformations.

Other realizations of the Heisenberg algebra (2.2) are of interest too. Here we will mention only one of them given in terms of the standard creation and annihilation operators a^{+j} and a_j and the Fock space. The standard basis in the Fock space is given by the vectors

$$|n_0, n_1, \ldots, n_{N-1}\rangle = (n_0! n_1! \ldots n_{N-1}!)^{-\frac{1}{2}} \sqcap_{k=0}^{N-1} (a^{+k})^{n_k} |0\rangle, \qquad n_k = 0, 1, 2 \ldots$$

where $a_j\,|0\rangle = 0$, $j = 0, 1, \ldots, N-1$ and $\langle 0|0\rangle = 1$.

Then

$$|\Psi\rangle = \sum_{n_k=0}^{\infty} A_{n_0, n_1, \ldots, n_{N-1}}\,|n_0, n_1, \ldots, n_{N-1}\rangle$$

and the constraint (2.5) takes the form

$$\left(-a_j a_k + C^l_{jk}(a^+) a_l\right) |\Psi\rangle = 0, \qquad j, k = 0, 1, \ldots, N-1. \tag{2.27}$$

This system of equations is equivalent to the infinite system of discrete equations for the coefficients $A_{n_0,\ldots,n_{N-1}}$ while $C^l_{jk}(a^+)$ obey QCS (2.13). Equations (2.27) define sort of coherent states which could be relevant to the theory of quantum deformations and its quasiclassical limit.

Finally, we note that several different 'quantization' schemes for the structures associated with the Frobenius manifolds, F-manifolds and coisotropic submanifolds have been proposed in [18–22]. A comparative analysis of these approaches and our scheme will be done elsewhere.

3. Isoassociative quantum deformations and oriented associativity equation

General quantum deformations described in the previous section contain as a subclass of deformations for which the classical associativity condition (1.4) is satisfied for all values of quantum deformation parameters. We will refer to such deformations as isoassociative quantum deformations by analogy with the isomonodromy and isospectral deformations. Formula (2.13) implies

Proposition 3.1. *Structure constants $C^l_{jk}(x)$ define isoassociative quantum deformations of associative algebra if they obey the equations*

$$C^m_{jk}C^n_{lm} - C^m_{kl}C^n_{jm} = 0, \tag{3.1}$$

$$\frac{\partial C^n_{jk}}{\partial x^l} - \frac{\partial C^n_{kl}}{\partial x^j} = 0. \tag{3.2}$$

In terms of the one-form Γ *(2.18) the system (3.1), (3.2) looks like*

$$\Gamma \wedge \Gamma = 0, \qquad d\Gamma = 0. \tag{3.3}$$

Another way to arrive at system (3.1), (3.2) consists in the treatment of $\hbar$ in all the above formulae beginning with (2.2) not as the fixed constant but as a variable parameter. In such interpretation the QCS (2.13) from the very beginning splits into two equations (3.1) and (3.2), the connection ∇_j from (2.20) becomes a pencil of flat torsionless connection discussed in [5, 6, 8, 9] and equations (2.24) coincide with the Dubrovin's linear system for flat coordinates [5, 6]. Thus, quantum deformations for which p_j and x^j are elements of the pencil of Heisenberg algebras (2.2) are of particular interest.

A way to deal with system (3.1), (3.2) is to solve first equations (3.2). They imply that

$$C^l_{jk} = \frac{\partial^2 \Phi^l}{\partial x^j \partial x^k}, \tag{3.4}$$

where $\Phi^l, l = 0, 1, \ldots, N-1$ are functions. Equation (3.1) then become

$$\frac{\partial^2 \Phi^m}{\partial x^j \partial x^k}\frac{\partial^2 \Phi^n}{\partial x^m \partial x^l} = \frac{\partial^2 \Phi^m}{\partial x^l \partial x^k}\frac{\partial^2 \Phi^n}{\partial x^m \partial x^j}. \tag{3.5}$$

System (3.5) has appeared first in [5] (Proposition 2.3) as the equation for the displacement vector. It has been rederived in a different context in [23] and has been called the oriented associativity equation there. In the form (3.3) it has appeared also in [24, 25]. In our approach it describes the isoassociative quantum deformation of the structure constants for a class of associative noncommutative algebras. For this class of deformations all operators f_{jk} have a simple generating 'function', namely

$$\hbar^2 f_{jk} = [p_{j,}[p_k, W]],$$

where

$$W = -\tfrac{1}{2}(x^m p_m)^2 + \Phi^m p_m.$$

In the theory of coisotropic deformations [14] the deformations given by equations (3.4), (3.5) constitute a subclass of all deformations. So, oriented associativity equation describes simultaneously both coisotropic and isoassociative quantum deformations. In other words, one of the characteristic features of the class of deformations given by formulae (3.1), (3.2) is that they remain unchanged in the process of 'quantization'. This means also that one can use both classical formulae [14] and the quantum one (previous section) to describe the properties of these deformations. For instance, it was shown in [14] that in the natural parametrization of the structure constants C^l_{jk} by the eigenvalues of the matrices C_j and in terms of canonical coordinates u^j the system (3.2) becomes the system of conditions for the commutativity of N hydrodynamical-type systems. At the same time, the functions Φ^n have a meaning of conserved densities for these hydrodynamical-type systems. All these results are valid for the isoassociative quantum deformations too.

The oriented associativity equation (3.5) admits a well-known reduction to a single superpotential F given by

$$\Phi^n = \eta^{nl}\frac{\partial F}{\partial x^l},$$

where η^{nl} is a constant metric. In this case equations (3.5) become the famous WDVV equation [3, 4]

$$\frac{\partial^3 F}{\partial x^j \partial x^k \partial x^s} \eta^{st} \frac{\partial^3 F}{\partial x^t \partial x^m \partial x^l} = \frac{\partial^3 F}{\partial x^l \partial x^k \partial x^s} \eta^{st} \frac{\partial^3 F}{\partial x^t \partial x^m \partial x^j}. \tag{3.6}$$

Thus, the WDVV equation also describes the isoassociative quantum deformations.

One more example of isoassociative quantum deformations is provided by the Riemann space with the flat Hessian metric

$g_{jk} = \frac{\partial^2 \Theta}{\partial x^j \partial x^k}$ considered in [26] (see also [27], proposition 5.10). In this case [26]

$$C^l_{jk} = \hbar \Gamma^l_{jk} = \hbar g^{lm} \frac{\partial^3 \Theta}{\partial x^j \partial x^k \partial x^m}, \tag{3.7}$$

equation (3.2) is satisfied identically and the associativity condition takes the form

$$\frac{\partial^3 \Theta}{\partial x^j \partial x^k \partial x^s} g^{st} \frac{\partial^3 \Theta}{\partial x^t \partial x^m \partial x^l} = \frac{\partial^3 \Theta}{\partial x^l \partial x^k \partial x^s} g^{st} \frac{\partial^3 \Theta}{\partial x^t \partial x^m \partial x^j}. \tag{3.8}$$

This equation represents a rather nontrivial single-field reduction of equation (3.5).

4. Weakly (non)associative quantum deformations

All coisotropic deformations are isoassociative by construction [14]. For their subclass described by the oriented associativity equation the CS is reduced to the system (3.1), (3.2). But, there is another subclass of coisotorpic deformations for which the exactness conditions (3.2) are not satisfied. For the finite-dimensional algebras such coisotropic deformations are described by the stationary dispersionless KP equation and other hydrodynamical-type systems [14] . In the infinite-dimensional case this type of deformations is described by the universal Whitham hierarchy of zero genus and, in particular, by the dispersionless KP hierarchy [12].

What is the quantum version of coisotropic deformations of such a type? One naturally expects that they will not be the isoassociative one. On the other hand, quantum deformations, for which equations (3.2) are not satisfied, are governed by equation (2.13) with a nice geometrical meaning even if they are not isoassociative.

All these suggest that the general quantum deformations defined by QCS (2.13) without the additional exactness constraint (3.2) should be of interest too. We will refer to such deformations as weakly associative or weakly nonassociative quantum deformations.

The first term is due to the fact that according to (2.10) and (2.12) for such deformations one has associativity for all values of quantum deformation parameters not on the operator level, i.e. not on the whole space H, but only on the smaller 'physical' subspace H_Γ. The second name reflects the fact that the defect of associativity

$$\alpha^n_{klj} \doteqdot C^m_{jk} C^n_{lm} - C^m_{kl} C^n_{jm} \tag{4.1}$$

for such deformations is given by

$$\alpha^n_{klj} = -\hbar \left(\frac{\partial C^n_{jk}}{\partial x^l} - \frac{\partial C^n_{kl}}{\partial x^j} \right). \tag{4.2}$$

So, for small $\hbar$ or slowly varying structure constants the defect of associativity is small.

For the matrix-valued two-form α_q with the matrix elements $(\alpha_q)^n_k \doteqdot \frac{1}{2} \alpha^n_{klj}\, \mathrm{d}x^l \wedge \mathrm{d}x^j$ one has

$$\alpha_q = -\hbar d\Gamma, \tag{4.3}$$

where Γ is defined in (2.18). One may refer to α_q also as a quantum anomaly of associativity. Note that for an algebra with unity element all elements α^n_{klj} with k or l or $j = 0$ vanish.

Geometrical interpretation of the QCS (2.13) provides us with numerous examples of weakly (non)associative quantum deformations. Any torsionless flat connection gives us such deformation for certain associative algebra. In the generic case, for instance, these deformations are given by formulae (2.22) and (2.23) for the semisimple and nilpotent algebras, respectively.

If there exists a metric g_{jk} compatible with the Christoffel symbols $\Gamma^l_{jk} = \hbar C^l_{jk}$ then the generic deformation of the structure constants is described by the formula

$$C^l_{jk} = \frac{1}{2}\hbar g^{nl}\left(\frac{\partial g_{nk}}{\partial x^j} + \frac{\partial g_{jn}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^n}\right), \tag{4.4}$$

where g_{jk} is an arbitrary flat metric. Particular choice of the metric gives us a specific deformation. For instance, for the diagonal flat metric $g_{jk} = \delta_{jk}H_j^2$ the weakly (non) associative deformations are defined by the solutions of the well-known Lame system which describes the orthogonal systems of coordinates in the N-dimensional Euclidean space (see e.g. [16]). For certain metrics, as, for example, for the Hessian metric considered at the end of the previous section, the quantum anomaly vanishes.

Other examples are provided by interpretation of the system (2.24) as the system of equations for the position vector in the affine differential geometry (see e.g. [28]). We note also the papers [29, 30] in which the equations describing the geometry of submanifolds for a flat space have been reduced to the WDVV-type equations.

Different types of (non)associative deformations is given by the quantum version of the coisotropic deformations of the finite-dimensional algebras studied in [14]. As an illustrative example we will consider here the three-dimensional ($N = 3$) algebra with the unity element. The nontrivial part of the table of multiplication is of the form

$$\begin{aligned}
&\mathbf{P}_1^2 = A\mathbf{P}_0 + B\mathbf{P}_1 + C\mathbf{P}_2,\\
&\mathbf{P}_1\mathbf{P}_2 = D\mathbf{P}_0 + E\mathbf{P}_1 + G\mathbf{P}_2,\\
&\mathbf{P}_2^2 = L\mathbf{P}_0 + M\mathbf{P}_1 + N\mathbf{P}_2.
\end{aligned} \tag{4.5}$$

As in paper [14] we consider the 'gauge' $B = 0, C = 1, G = 0$. The QCS system (2.13) in this case assumes the form

$$\begin{aligned}
&A + N - E = 0,\\
&\hbar A_{x_2} - \hbar D_{x_1} + L - EA = 0,\\
&-\hbar E_{x_1} + M - D = 0,\\
&\hbar D_{x_2} - \hbar L_{x_1} + ED - MA - ND = 0,\\
&\hbar E_{x_2} - \hbar M_{x_1} + E^2 - L - NE = 0,\\
&-\hbar N_{x_1} + D - M = 0,
\end{aligned} \tag{4.6}$$

where $A_{x_j} \doteqdot \frac{\partial A}{\partial x^j}$ etc. This system of equations implies that

$$\begin{aligned}
&E = \frac{1}{2}A + \frac{3}{4}\epsilon, \qquad N = -\frac{1}{2}A + \frac{3}{4}\varepsilon,\\
&L = \frac{1}{2}A^2 + \frac{3}{4}\epsilon A - \hbar A_{x_2} + \hbar D_{x_1},\\
&M = D + \frac{\hbar}{2}A_{x_1}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&A_{x_2} - \frac{4}{3}D_{x_1} + \epsilon_{x_2} - \frac{\hbar}{3}A_{x_1x_1} = 0,\\
&D_{x_2} - \frac{3}{4}(A^2)_{x_1} - \frac{3}{4}\epsilon A_{x_1} + \hbar A_{x_1x_2} - \hbar D_{x_1x_1} = 0,
\end{aligned} \tag{4.8}$$

where $\epsilon(x_2)$ is an arbitrary function. Eliminating D from system (4.8) ,one obtains the equation

$$A_{x_2x_2} - \epsilon A_{x_1x_1} - (A^2)_{x_1x_1} + \frac{\hbar^2}{3} A_{x_1x_1x_1x_1} + \epsilon_{x_2x_2} = 0. \tag{4.9}$$

At ϵ = const it is the well-known Boussinesq equation which describes surface waves (see e.g. [31]). This equation is integrable by the inverse scattering transform method [32] similar to the famous Korteweg-de Vries and KP equations (see e.g. [33–35]).

Equations (4.8) imply the existence of the function F such that

$$A = -\epsilon - 2F_{x_1x_1}, \qquad D = -\frac{3}{2}F_{x_1x_2} + \frac{\hbar}{2}F_{x_1x_1x_1}. \tag{4.10}$$

In terms of function F, system (4.8) or equation (4.9) becomes

$$F_{x_2x_2} - \epsilon F_{x_1x_1} + \frac{1}{2}\left(\epsilon + 2F_{x_1x_1}\right)^2 + \frac{\hbar^2}{3}F_{x_1x_1x_1x_1} = 0. \tag{4.11}$$

The function τ defined by $F = \log\tau$ is the τ- function for the Boussinesq equation (4.9) and equation (4.11) is the Hirota equation to it (at $\epsilon = 0$ see e.g. [36]).

Any solution of the Boussinesq equation (4.9) or the Hirota equation (4.11) provides us with the weakly (non)associative quantum deformation of the algebra (4.5) with the structure constants given by formulae (4.7), (4.10). The quantum anomaly α_q (4.3) for these Boussinesq deformations is of the form

$$\alpha_q = \hbar \begin{pmatrix} 0 & \frac{1}{2}A_{x_2} + \frac{\hbar}{2}A_{x_1x_1} & \frac{1}{2}(A^2)_{x_1} \\ 0 & -A_{x_1} & -\frac{1}{2}A_{x_2} - \frac{\hbar}{2}A_{x_1x_1} \\ 0 & 0 & A_{x_1} \end{pmatrix} \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{4.12}$$

To present a simple concrete example of deformation for the algebra (4.5) we consider the following polynomial solution

$$F = \alpha(x^1)^2 + \beta x^1x^2 + (\alpha\epsilon - 4\alpha^2)(x^2)^2 + \gamma(x^1)^2x^2 + \tfrac{1}{3}\gamma(\epsilon - 8\alpha)(x^2)^3 - \tfrac{2}{3}\gamma^2(x^2)^4 \tag{4.13}$$

of the Hirota equation (4.11) with ϵ=const where α, β, γ are arbitrary constants. This solution defines via (4.7) and (4.10) the following weakly (non)associative deformation of the structure constants:

$$\begin{aligned} &A = -4\alpha - 4\gamma x^2, \qquad B = 0, \qquad C = 1, \\ &D = -\tfrac{3}{2}\beta - 3\gamma x^1, \qquad E = -2\alpha - 2\gamma x^2 + \tfrac{3}{4}\epsilon, \qquad G = 0, \\ &L = 8(\alpha + \gamma x^2)^2 - 3\epsilon(\alpha + \gamma x^2) + \hbar\gamma, \qquad M = -\tfrac{3}{2}\beta - 3\gamma x^1, \\ &N = 2\alpha + 2\gamma x^2 + \tfrac{3}{4}\epsilon. \end{aligned} \tag{4.14}$$

For this deformation the quantum anomaly is given by

$$\alpha_q = 2\hbar\gamma \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{4.15}$$

General formulae (2.5) and (2.24) provide us with the auxiliary linear problems for the Boussinesq equation. It is easy to show that equations (2.5) for the algebra (4.5) with $B = G = 0, C = 1$ are equivalent to the following two equations

$$\left(p_1^3 - \left(\frac{3}{2}A + \frac{3}{4}\epsilon\right)p_1 - \left(D + \hbar A_{x_1}\right)p_0\right)|\Psi\rangle = 0, \tag{4.16}$$

$$\left(p_2 - p_1^2 + Ap_0\right)|\Psi\rangle = 0. \tag{4.17}$$

In the coordinate representation these equations look like

$$\hbar^3\Psi_{x_1x_1x_1} + \frac{3}{2}\hbar\left(u - \frac{\epsilon}{2}\right)\Psi_{x_1} + w\Psi = 0, \tag{4.18}$$

$$\hbar\Psi_{x_2} - \hbar^2\Psi_{x_1x_1} - u\Psi = 0, \tag{4.19}$$

where $u = -A$ and $w = -D - \hbar A_{x_1}$. Equations (4.18)–(4.19) are the well-known auxiliary linear problems for the Boussinesq equation at zero value of the spectral parameter [32–34].

Another set of linear problems can be obtained from conditions (2.20). For the Boussinesq algebra (4.5) the matrices C_1 and C_2 are

$$C_1 = \begin{pmatrix} 0 & A & D \\ 1 & 0 & \frac{1}{2}A + \frac{3}{4}\epsilon \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.20}$$

$$C_2 = \begin{pmatrix} 0 & D & \frac{1}{2}A^2 - \hbar A_{x_2} + \hbar D_{x_1} + \frac{3}{4}\epsilon A \\ 0 & \frac{1}{2}A + \frac{3}{4}\epsilon & D + \frac{\hbar}{2}A_{x_1} \\ 1 & 0 & -\frac{1}{2}A + \frac{3}{4}\epsilon \end{pmatrix}. \tag{4.21}$$

The commutativity condition (2.20) for the connection $\nabla_j = \hbar\frac{\partial}{\partial x^j} + C_j$, $J = 1, 2$ represents the compatibility condition for the linear problems

$$\left(\hbar\frac{\partial}{\partial x^1} + C_1\right)\varphi = 0, \qquad \left(\hbar\frac{\partial}{\partial x^2} + C_2\right)\varphi = 0, \tag{4.22}$$

where φ is the column $\varphi = (\varphi_{1,}\varphi_2, \varphi_3)^T$. Equations (2.20) for C_1 and C_2 given by (4.20) and (4.21) are equivalent to equations (4.7), (4.8), i.e. to the Boussinesq equation. So, equations (4.22) represent the auxiliary matrix linear problems for the Boussinesq equation. Equations (4.22) imply the scalar equations

$$\begin{aligned} &\hbar^3\varphi_{3,x_1x_1x_1} - \hbar\left(\frac{3}{2}A + \frac{3}{4}\epsilon\right)\varphi_{3,x_1} + \left(D + \frac{\hbar}{2}A_{x_1}\right)\varphi_3 = 0, \\ &\hbar\varphi_{3,x_2} + \hbar^2\varphi_{3,x_1x_1} - A\varphi_3 = 0. \end{aligned} \tag{4.23}$$

Equations (4.23) are formally adjoint to equations (4.18)–(4.19) and their compatibility condition gives rise to the same Boussinesq equation (4.9).

We would like to note that the 'zero curvature' representation $\left[\frac{\partial}{\partial x^1} + U, \frac{\partial}{\partial x^2} + V\right] = 0$ with the matrix-valued functions U and V is quite common in the theory of the (1+1)-dimensional integrable systems (see e.g. [33–35]). The particular representation of the form (4.22) is of interest at least by two reasons. First, for instance, for the Boussinesq equation the elements of the matrices C_1 and C_2 (4.20), (4.21) really coincide with the components of the Christoffel symbol. Second, in such a representation the elements of C_1 and C_2 are nothing but the structure constants of the deformed associative algebra (4.5).

All the above formulae for the Boussinesq quantum deformations in the formal limit $\hbar \to 0$ (with $\Psi = \exp\left(\frac{S}{\hbar}\right)$) are reduced to those for the coisotropic deformations of the same algebra (4.5) which are described by the stationary dispersionless KP equation [14].

Finally, we note that eliminating $F_{x_1x_1}$ from the Hirota equation (4.11) with the use of its differential consequences, one obtains the equation

$$F_{x_2x_2x_2}F_{x_1x_1x_1} - F_{x_2x_2x_1}F_{x_1x_1x_2} = \frac{\hbar^2}{3}\left(F_{x_1x_1x_2}F_{x_1x_1x_1x_1x_1} - F_{x_1x_1x_1}F_{x_1x_1x_1x_1x_2}\right)$$

which represents the 'quantum' version of Witten's equation [3]

$$F_{x_2x_2x_2}F_{x_1x_1x_1} - F_{x_2x_2x_1}F_{x_1x_1x_2} = 0,$$

i.e. equation (3.6) for the two-dimensional algebra without unity element and the metric $\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

5. Quantum deformations of the infinite-dimensional algebra and KP hierarchy

We will consider an infinite-dimensional algebra of polynomials generated by a single element. In the so-called Faa' di Bruno basis the structure constants of this algebra have the form [12]

$$C^l_{jk} = \delta^l_{j+k} + H^k_{j-l} + H^j_{k-l}, \qquad j,k,l = 0,1,2\ldots, \tag{5.1}$$

where $H^k_l = 0$ at $l \leqslant 0$ and $H^0_l = 0$. Coisotropic deformations of the structure constants (5.1) have been studied in [12]. It was shown that they are described by the dispersionless KP hierarchy.

Here we will discuss the quantum deformations of the same set (5.1) of the structure constants.

Proposition 5.1. *For the structure constants (5.1) the QCS (2.13) is equivalent to the system*

$$\hbar\frac{\partial H^k_j}{\partial x^l} + H^l_{j+k} + H^k_{j+l} - H^{k+l}_j + \sum_{n=1}^{j-1} H^k_{j-n}H^l_n - \sum_{n=1}^{l-1} H^k_{l-n}H^n_j - \sum_{n=1}^{k-1} H^l_{k-n}H^n_j = 0,$$
$$j,k,l = 0,1,2,\ldots. \tag{5.2}$$

Proof. Substitution of (5.1) into (2.13) gives

$$\hbar\frac{\partial H^k_{j-m}}{\partial x^l} + \hbar\frac{\partial H^j_{k-m}}{\partial x^l} - \hbar\frac{\partial H^k_{l-m}}{\partial x^j} - \hbar\frac{\partial H^l_{k-m}}{\partial x^j}$$
$$+\sum_{n=0}^{\infty}\left(\delta^n_{j+k} + H^k_{j-n} + H^j_{k-n}\right)\left(\delta^m_{n+l} + H^l_{n-m} + H^n_{l-m}\right)$$
$$-\sum_{n=0}^{\infty}\left(\delta^n_{l+k} + H^k_{l-n} + H^l_{k-n}\right)\left(\delta^m_{n+j} + H^j_{n-m} + H^n_{j-m}\right) = 0. \tag{5.3}$$

At $j > m, k < m, l < m$ using the identity

$$\sum_{p=n-1}^{k-1} H^j_{k-p}H^m_{p-n} = \sum_{p=n-1}^{k-1} H^m_{k-p}H^j_{p-n},$$

one obtains equation (5.2) with the substitution $j \to j-m$. At $m > j, m > l$ and $m < k$ equation (5.3) is reduced to

$$\frac{\partial H^j_k}{\partial x^l} - \frac{\partial H^l_k}{\partial x^j} = 0. \tag{5.4}$$

It is easy to see that equation (5.2) directly implies (5.4) due to the symmetry of the nondifferential part in the indices k and l. An analysis of all other choices of indices in (5.3) shows that the resulting equations are all equivalent to (5.2). □

Solutions of the QCS (5.2) provide us with the quantum deformation of the polynomial algebra in the Faa' di Bruno basis. In general, these deformations are the weakly (non)associative one and the quantum anomaly is given by

$$\alpha^n_{klj} = \hbar\left(\frac{\partial H^k_{l-n}}{\partial x^j} - \frac{\partial H^k_{j-n}}{\partial x^l}\right)$$

or

$$\alpha_q = -\hbar\, \mathrm{d}A, \tag{5.5}$$

where the matrix-valued one -form A has elements $(A)^n_k = \sum_{l=n+1}^{\infty} H^k_{l-n}\, \mathrm{d}x^l$. At $\hbar \to 0$ the QCS (5.2) converts into the classical associativity condition for the structure constants (5.1) [12].

We note that for the first time the system (5.2) has been derived in [37] within a different context as the component-wise version of the central system for the currents associated with the KP hierarchy. It was shown in [37] that it encodes a complete algebraic information about the KP hierarchy. We will demonstrate this in a little bit different manner.

Similar to the coisotropic case [12] there are, at least, two ways to decode information contained in the QCS (5.2). First approach is to choose first an appropriate parametrization of H^j_k. As in the classical case [12] we introduce the functions u, v and w defined by the formulae

$$H^1_1 = -\tfrac{1}{2}u, \qquad H^1_2 = -\tfrac{1}{3}v, \qquad H^1_3 = -\tfrac{1}{4}w + \tfrac{1}{8}u^2. \tag{5.6}$$

From the QCS (5.2) one obtains

$$H^2_1 = 2H^1_2 + \hbar \frac{\partial H^1_1}{\partial x^1},$$
$$H^3_1 = 3H^1_3 + \hbar \frac{\partial \left(H^1_2 + H^2_1\right)}{\partial x^1},$$
$$H^2_2 = -\hbar \frac{\partial H^1_1}{\partial x^2} - H^1_3 + H^3_1 + \left(H^1_1\right)^2.$$

Hence

$$\begin{aligned} H^2_1 &= -\frac{2}{3}v - \frac{\hbar}{2}u_{x_1}, \\ H^3_1 &= -\frac{3}{4}w + \frac{3}{8}u^2 - \hbar v_{x_1} - \frac{\hbar^2}{2}u_{x_1x_1}, \\ H^2_2 &= \frac{1}{2}u^2 - \frac{1}{2}w + \frac{\hbar}{2}u_{x_2} - \hbar v_{x_1} - \frac{\hbar^2}{2}u_{x_1x_1}. \end{aligned} \tag{5.7}$$

Substituting these expressions into the first exactness conditions (5.4), i.e.

$$\frac{\partial H^1_1}{\partial x^2} - \frac{\partial H^2_1}{\partial x^1} = 0, \frac{\partial H^1_2}{\partial x^2} - \frac{\partial H^2_2}{\partial x^1} = 0, \qquad \frac{\partial H^1_1}{\partial x^3} - \frac{\partial H^3_1}{\partial x^1} = 0 \tag{5.8}$$

and eliminating w, one obtains the equations

$$u_{x_3} - \frac{\hbar^2}{4}u_{x_1x_1x_1} - \frac{3}{4}(u^2)_{x_1} - \varphi_{x_2} = 0, \qquad u_{x_2} - \frac{4}{3}\varphi_{x_1} = 0, \tag{5.9}$$

where $\varphi = v + \frac{3}{4}\hbar u_{x_1}$. This is the famous Kadomtsev–Petviashvili equation (see e.g. [33–35]). Using higher equations (5.2) and (5.4), one in a similar manner obtains the higher KP equations and the whole KP hierarchy.

In the limit $\hbar \to 0$ equation (5.9) is reduced to the dispersionless KP equation while at the stationary case $u_{x_3} = 0$ one obtains the Boussinesq equation (4.9) at $\epsilon = 0$ with $A = -u, \varphi = -D - \frac{\hbar}{4}A_{x_1}$.

Another way to deal with the QCS (5.2) is to solve first all exactness conditions. One of them is given by (5.4). It implies the existence of the functions F_k such that

$$H^j_k = \frac{\partial F_k}{\partial x^j}, \qquad j, k = 1, 2, 3 \ldots \tag{5.10}$$

The system (5.2), in addition, contains another exactness-type condition. Indeed, as it was shown in [37], equations (5.2) lead to the following constraint system

$$\hbar\frac{\partial}{\partial x^j}\left(\sum_{k=1}^{n-1}H_{n-k}^k\right)+nH_n^j=H_n^j+\sum_{k=1}^{n-1}\left(H_k^{j+n-k}-H_{j+n-k}^k\right)+\sum_{l=1}^{j-1}\sum_{k=1}^{n-1}H_l^{n-k}H_k^{j-l}. \tag{5.11}$$

The rhs of (5.11) is symmetric in the indices j and n. Hence

$$\hbar\frac{\partial}{\partial x^j}\left(\sum_{k=1}^{n-1}H_{n-k}^k\right)+nH_n^j=\hbar\frac{\partial}{\partial x^n}\left(\sum_{k=1}^{j-1}H_{j-k}^k\right)+jH_j^n. \tag{5.12}$$

Substitution of (5.10) into equations (5.12) gives the exactness conditions

$$\frac{\partial}{\partial x^j}\left(nF_n+\hbar\sum_{k=1}^{n-1}\frac{\partial F_{n-k}}{\partial x^k}\right)=\frac{\partial}{\partial x^n}\left(jF_j+\hbar\sum_{k=1}^{j-1}\frac{\partial F_{j-k}}{\partial x^k}\right),\qquad j,n=1,2,3\ldots \tag{5.13}$$

Proposition 5.2.

$$H_k^j=\frac{1}{\hbar}\mathrm{P}_k(-\hbar\widetilde{\partial})F_{x_j},\qquad j,k=1,2,3,\ldots \tag{5.14}$$

where $P_k(-\hbar\widetilde{\partial})\doteqdot P_k\left(-\hbar\frac{\partial}{\partial x^1},-\frac{1}{2}\hbar\frac{\partial}{\partial x^2},-\frac{1}{3}\hbar\frac{\partial}{\partial x^3},\ldots\right)$ *and* $P_k(t_1,t_2,t_3,\ldots)$ *are Schur polynomials.*

Proof. Equations (5.13) imply the existence of a function F such that

$$jF_j+\hbar\sum_{k=1}^{j-1}\frac{\partial F_{j-k}}{\partial x^k}=-F_{x_j}. \tag{5.15}$$

Resolving (5.15) recurrently, one obtains

$$\begin{aligned}F_1&=-F_{x_1},\\2F_2&=-F_{x_2}+\hbar F_{x_1x_1},\\3F_3&=-F_{x_3}+\tfrac{3}{2}\hbar F_{x_1x_2}-\tfrac{1}{2}\hbar^2F_{x_1x_1x_1}\end{aligned}$$

and so on. The compact form of these relations is $F_k=\frac{1}{\hbar}P_k(-\hbar\widetilde{\partial})F$ where Schur polynomials are defined, as usual, by the generating formula $\exp\left(\sum_{k=1}^{\infty}\lambda^kt_k\right)=\sum_{k=0}^{\infty}\lambda^kP_k(\mathbf{t})$. Then, in virtue of (5.10), one has (5.14). □

Substitution of the expressions (5.14) for H_k^j into the QCS (5.2) gives the infinite system of differential equations, bilinear in F. The simplest of them is

$$\frac{4}{3}F_{x_1x_3}-\frac{\hbar^2}{3}F_{x_1x_1x_1x_1}-2\left(F_{x_1x_1}\right)^2-F_{x_2x_2}=0. \tag{5.16}$$

In terms of the function $\tau=\exp F$ the above equation and equations (5.2) with

$$H_k^j=\frac{1}{\hbar}\mathrm{P}_k(-\hbar\widetilde{\partial})\frac{\tau_{x_j}}{\tau},\qquad j,k=1,2,3,\ldots \tag{5.17}$$

are nothing but the famous bilinear Hirota equations for the KP hierarchy (see e.g. [34–36]). Hence, the function τ is the celebrated KP τ-function.

Thus, any KP τ-function defines weakly (non)associative quantum deformations of the structure constants (5.1) for the infinite-dimensional algebra by formula (5.17) and Hirota bilinear equations. Quantum anomaly for these deformations is given by (5.5) with

$$(A)_k^n=\frac{1}{\hbar}\sum_{l=n+1}^{\infty}P_{l-n}(-\hbar\widetilde{\partial})\frac{\tau_{x_k}}{\tau}\,\mathrm{d}x^l. \tag{5.18}$$

At last, in the limit $\hbar \to 0$ all the above formulae are reduced to those for coisotropic deformations [12]. We emphasize that quantum and isotropic deformations represent different deformations of the same structure constants (5.1).

For concrete solutions of the KP hierarchy certain structure constants may remain undeformed and components of quantum anomaly may vanish. For example, the function F given by (4.14) with $\epsilon = 0$ is the solution of equation (5.16) too. For this solution all H_k^j with $j \geqslant 3$ vanish as well as $A_k^n = 0$ for $n, k \geqslant 3$. One soliton solution of the KP equation corresponds to $\tau = 1 + \exp[k(x^1 + px^2 + qx^3)]$ where $q = \frac{\hbar^2}{4}k^2 + \frac{3}{4}p^2$ and k,p are arbitrary constants (see e.g. [33–36]). For this soliton deformation $H_k^j = 0$ at $j \geqslant 4$ and $A_k^n = 0$ for $k, n \geqslant 4$.

A quasi-triangular structure of the constants (5.1) allows us to rewrite equations (2.5) in the equivalent form

$$\left(p_n - p_1^n - \sum_{m=1}^{n-2} u_{nm}(x)\, p_1^m - u_{n0}p_0\right)|\Psi\rangle = 0, \qquad n = 1, 2, 3, \ldots \tag{5.19}$$

where the coefficients u_{nm} are the certain functions of H_k^j. For example, $u_{20} = -2H_1^1$, $u_{31} = -3H_1^1$, $u_{30} = H_2^1 + H_1^2 + 2\frac{\partial H_1^1}{\partial x^1}$.

In the coordinate representation equations (2.24) due to (2.25) take the form

$$-\hbar^2\frac{\partial^2\widetilde{\Psi}}{\partial x^j\partial x^k} + \hbar\frac{\partial\widetilde{\Psi}}{\partial x^{j+k}} + \hbar\sum_{l=1}^{j-1} H_{j-l}^k\frac{\partial\widetilde{\Psi}}{\partial x^l} + \hbar\sum_{l=1}^{k-1} H_{k-l}^j\frac{\partial\widetilde{\Psi}}{\partial x^l} + \left(H_k^j + H_j^k\right)\widetilde{\Psi} = 0, \quad j, k = 1, 2, 3, \ldots \tag{5.20}$$

The system of linear equations (5.20) is equivalent to the standard set of auxiliary linear problems for the KP hierarchy

$$\hbar\frac{\partial\widetilde{\Psi}}{\partial x^n} = \hbar^n\frac{\partial^n\widetilde{\Psi}}{(\partial x^1)^n} + \sum_{m=0}^{n-2}\hbar^m u_{nm}(x)\frac{\partial^m\widetilde{\Psi}}{(\partial x^1)^m} \tag{5.21}$$

that is the coordinate representation of equations (5.19).

To get a standard form of the above formulae with a spectral parameter z one considers a formal Laurent series $H^{(j)}(x, z) \doteqdot \sum_1^\infty z^{-k}H_k^j$. In virtue of (5.15) one has

$$H^{(j)} = \frac{1}{\hbar}\frac{\partial}{\partial x^j}\left\{\left(\exp\left(-\hbar\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial x^n}\right) - 1\right)F\right\} = \frac{1}{\hbar}\frac{\partial}{\partial x^j}\log\left(\frac{\tau\left(x - [z^{-1}]\right)}{\tau(x)}\right), \tag{5.22}$$

where $x - [z^{-1}] \doteqdot \left(x^1 - \frac{1}{z}, x^2 - \frac{1}{2z^2}, x^3 - \frac{1}{3z^3}, \ldots\right)$ is the Miwa shift [36]. Introducing the wavefunction χ by $H^{(j)} \doteqdot \frac{1}{\hbar}\frac{\partial \log\chi}{\partial x^j}$, one obtains

$$\chi(x, z) = \frac{\tau(x - [z^{-1}])}{\tau(x)} \tag{5.23}$$

that reproduces the standard form of the dressed KP wavefunction

$$\widetilde{\Psi}(x, z) = \exp\left(\sum_{n=1}^{\infty} z^n x^n\right)\chi(x, z) = \exp\left(\sum_{n=1}^{\infty} z^n x^n\right)\frac{\tau(x - [z^{-1}])}{\tau(x)} \tag{5.24}$$

in terms of the τ-function [36]. For more details see [37].

It is well known that the stationary reductions of the KP hierarchy give rise to the Gelfand–Dickey hierarchies (see e.g. [33–35]). At the same time one can show that the stationarity constraint $\frac{\partial H_k^j}{\partial x^N} = 0$ converts the infinite-dimensional polynomial algebra into the finite-dimensional one. So, stationary solutions of the KP hierarchy provide us with the weakly (non)associative quantum deformations of the finite-dimensional algebras. The Boussinesq deformation (4.8)–(4.12) is the simplest example. For the general Gelfand–Dickey case see also [18].

Finally, we would like to note that the quantum deformations of algebras obtained by the process of gluening [12] of N algebras of the type (5.1) are described by the N-component KP hierarchy.

At last, in the limit $\hbar \to 0$ all the above formulae are reduced to those for coisotropic deformations [12]. We emphasize that quantum and isotropic deformations represent different deformations of the same structure constants (5.1).

6. Conclusion

The approach presented in the paper can be extended in different directions. For instance, the basic idea of identification of the elements $\mathbf{P}_j$ of the basis and deformations parameters x^j with the elements of the Heisenberg algebra can be applied to other types of algebras.

We note also that formula (2.10) gives a simple realization for the previously discussed idea of geometrical interpretation of the associator for an algebra as a curvature tensor (see e.g. [38]). It suggests a natural generalization of quantum deformations to nonassociative algebras.

Acknowledgments

The author is very grateful to Franco Magri for numerous fruitful discussions.

References

[1] Gerstenhaber M 1964 On the deformation of rings and algebras *Ann. Math.* **79** 59–103
[2] Gerstenhaber M 1966 On the deformation of rings and algebras II *Ann. Math.* **84** 1–19
[3] Witten E 1990 On the structure of topological phase of two-dimensional gravity *Nucl. Phys.* B **340** 281–332
[4] Dijkgraaf R, Verlinde H and Verlinde E 1991 Topological strings in $d < 1$ *Nucl. Phys.* B **352** 59–86
[5] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89
[6] Dubrovin B 1996 Geometry of 2D topological field theories *Lecture Notes in Math.* vol 1620 (Berlin: Springer) pp 120–348
[7] Hertling C and Manin Y I 1999 Weak Frobenius manifolds *Int. Math. Res. Not.* **6** 277–86
[8] Manin Y I 2005 F-manifolds with flat structure and Dubrovin's duality *Adv. Math.* **198** 5–26
[9] Manin Y I 1999 *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces* (Providence, RI: American Mathematical Society)
[10] Hertling C 2002 *Frobenius Manifolds and Moduli Spaces for Singularities* (Cambridge: Cambridge University Press)
[11] Hertling C and Marcoli M (ed) 2004 *Frobenius Manifolds, Quantum Cohomology and Singularities (Aspects Math. E* vol 36*)* (Wiesbaden: Friedr. Vieweg & Sohn)
[12] Konopelchenko B G and Magri F 2007 Coisotropic deformations of associative algebras and dispersionless integrable hierarchies *Commun. Math. Phys.* **274** 627–58
[13] Konopelchenko B G and Magri F 2007 Dispersionless integrable equations as coisotropic deformations: extensions and reductions *Theor. Math. Phys.* **151** 803–19
[14] Konopelchenko B G and Magri F Coisotropic deformations of associative algebras and integrable systems (at press)
[15] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)
[16] Eisenhart L P 1926 *Riemann Geometry* (Princeton, NJ: Princeton University Press)
[17] Willmore T J 1993 *Riemann Geometry* (New York: Oxford)
[18] Givental A 2003 A_{n-1} singularities and nKdV hierarchies *Mosc. Math. J.* **3** 475–505
[19] Coates T and Givental A 2007 Quantum Riemann-Roch, Lefschetz and Serre *Ann. Math.* **165** 15–53

[20] Givental A 2001 Gromov-Witten invariants and quantization of quadratic Hamiltonians *Mosc. Math. J.* **1** 551–68

[21] Eliashberg Y 2007 Symplectic field theory and its applications *Plenary Lectures and Ceremonies, ICM 2006, European Math. Soc.* pp 217–246

[22] Cattaneo A S and Felder G 2007 Relative formality theorem and quantization of coisotropic manifolds *Adv. Math.* **208** 521–48

[23] Losev A and Manin Y I 2004 Extended modular operads *Frobenius Manifolds, Quantum Cohomology and Singularities (Aspects of Math. E* vol 36*)* ed C Hertling and M M Marcoli (Wiesbaden: Vieweg) pp 181–211

[24] Dubrovin B 1993 Geometry and integrability of topological-antitopological fusion *Commun. Math. Phys.* **152** 539–64

[25] Givental A and Kim B 1995 Quantum cohomology of flag manifolds and Toda lattices *Commun. Math. Phys.* **168** 609–42

[26] Kito H 1999 On Hessian structures of the Euclidean space and hyperbolic space *Osaka J. Math.* **36** 51–62

[27] Dubrovin B 2004 On almost duality for Frobenius manifolds *Geometry, Topology and Mathematical Physics Am. Math. Soc. Trans.* vol 212*)* (Providence, RI: American Mathematical Society) 75–132

[28] Nomizu K and Sasaki T 1994 *Affine Differential Geometry* (Cambridge: Cambridge University Press)

[29] Ferapontov E V 2004 Hypersurfaces with flat centroaffine metric and equations of associativity *Geom. Dedicata* **103** 33–49

[30] Mokhov O T 2007 Theory of submanifolds, associativity equations in 2D topological quantum field theory and Frobenius manifolds *Theor. Math. Phys.* **152** 1183–90

[31] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley Interscience)

[32] Zakharov V E 1974 On stochastization of one-dimensional chain of nonlinear oscillators *Sov. Phys. JETP* **35** 908–14

[33] Novikov S P, Manakov S V, Pitaevski L P and Zakharov V E 1984 *Theory of Solitons. The Inverse Problem Method* (New York: Plenum)

[34] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[35] Konopelchenko B G 1992 *Introduction to Multidimensional Integrable Equations* (New York: Plenum)

[36] Jimbo M and Miwa T 1983 Solitons and infinite-dimensional Lie algebras *Publ. Res. Inst. Math. Sci. Kyoto* **19** 943–1001

[37] Falgui G, Magri F and Pedroni M 1998 Bihamiltonian geometry, Darboux covering and linearization of the KP hierarchy *Commun. Math. Phys.* **197** 303–24

Casati P, Falqui G, Magri F and Pedroni M 1996 The KP theory revisited: IV. KP equations, dual KP equations, Baker-Akhiezer and τ functions, Preprint SISSA/5/96/FM, Trieste, Italy

[38] Ionescu L M 2003 Nonassociative algebras: a framework for differential geometry *Int. J. Math. Math. Sci.* **60** 3777–95